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SPHERICAL WAVES OF SPIN 1 PARTICLE IN ANTI DE SITTER SPACE-TIME

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Three possible techniques to deal with a vector particle in the anti de Sitter cosmological model are viewed: Duffin – Kemmer – Petiau matrix formalism based on the general tetrad recipe, group theory 5-dimensional approach based on the symmetry group $SO(3,2)$, and a tetrad form of Maxwell equations in complex Riemann – Silberstein – Majorana – Oppenheimer representation. In the first part, a spin 1 massive field is considered in static coordinates of the anti de Sitter space-time in tetrad-based approach. The complete set of spherical wave solutions with quantum numbers (ϵ, j, m, l) is constructed; angular dependence in wave functions is described with the help of Wigner functions. The energy quantization rule has been found. Transition to massless case of electromagnetic field is specified, and electromagnetic solutions in Lorentz gauge have been constructed. In the second part, the problem of a particle with spin 1 is considered on the base of 5-dimensional wave equation specified in the same static coordinates. In the third part, a rarely used approach, based on tetrad form of Maxwell equations in complex representation is examined in the anti de Sitter model.

Keywords: spin 1 field, anti de Sitter space, Duffin – Kemmer – Petiau formalism, $SO(3,2)$ group, 5-dimensional equation, Majorana – Oppenheimer representation,

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1 Introduction

Examining fundamental particle fields on the background of expanding universe, in particular de Sitter and anti de Sitter models, has a long history; special value of these geometries consists in their simplicity and high symmetry groups underlying them which makes us to believe in existence of exact analytical treatment for some fundamental problems of classical and quantum field theory in these curved spaces. In particular, there exist special representations for fundamental wave equations, Dirac's and Maxwell's, which are explicitly invariant under symmetry groups $SO(4,1)$ and $SO(3,2)$ for these models. In the most of the literature, when dealing with a spin 1 field in de Sitter models they use group theory approach. Many of the most important references are given below (it is not exhaustive bibliography, which should be enormous):

Dirac [1, 2], Schrödinger [3, 4], Lubanski–Rosenfeld [5], Goto [6], Ikeda [7], Nachtmann [8], Chernikov–Tagirov [9], Geheniau–Schombblond[10], Borner–Durr [11], Tugov [12], Fushchych–Krivsky [13], Chevalier [15], Castagnino [16, 19], Vidal [17], Adler [18], Schnirman–Oliveira [20], Tagirov [21], Riordan [22], Pestov–Chernikov–Shavoxina [23], Candelas–Raine [24], Schombblond–Spindel [25, 26], Dowker–Critchley [27], Avis–Isham–Storey [28], Brugarino [29], Fang–Fronsdal

[30], Angelopoulos et al [31], Burges [32], Deser-Nepomechie [33], Dullemond-Beveren [34], Gazeau [35], Allen [36], Fefferman-Graham [37], Flato-Fronsdal-Gazeau [38], Allen-Jacobson [39], Allen-Folacci [40], Sanchez [41], Pathinayake-Vilenkin-Allen [42], Gazeau-Hans [43], Bros-Gazeau-Moschella [44], Takook [45], Pol'shin [46, 47, 48], Gazeau-Takook [49], Takook [50], Deser-Waldron [51, 52], Spradlin-Strominger-Volovich [53], Cai-Myung-Zhang [54], Garidi-Huguet-Renaud [55], Rouhani-Takook [56], Behroozi et al [57], Huguet-Queva-Renaud [58], Garidi et al [59], Huguet-Queva-Renaud [60], Dehghani et al [61], Moradi-Rouhani-Takook [62], Faci et al [63]

The interest to exact solutions of wave equations for particles with different spins in de Sitter space was greatly increased in connection with Hawking radiation [64, 65, 66]. De Sitter cosmological model admits exact treatment in contrast to black hole space-time geometry:

Lohiya-Panchapakesan [67, 68], Khanal-Panchapakesan [69, 70], Khanal [71, 73], Hawking-Page [72], Chandrasekhar [147], Otchik [74], Motolla [75], Bogush-Otchik-Red'kov [76], Mishima-Nakayama [77], Polarski [78], Suzuki-Takasugi [79], Suzuki-Takasugi-Umetzu [80, 81, 82]

The case of anti de Sitter space seems to be less examined though it is also very interesting because of its topological properties. Any energy spectrum must be discreet, besides this geometry admits elliptical interpretation and physical manifestation of that in cosmological context is also of great importance. In the context of Hawking effect the most investigators used the Newman-Penrose formalism [132], [134], [147], [148].

Turning to the case of vector particle, we should note that many years ago a matrix Duffin - Kemmer - Petiau formalism was developed to treat a spin 1 field, it has a long and rich history inseparably linked with description of photons and mesons:

De Broglie [83, 84], De Broglie-Winter [85], Petiau [86], Proca [87, 88], Duffin [89], Kemmer [90, 91], Bhabha [92], Belinfante [93, 94], Sakata-Taketani [95], Tonnelat [96], Schrödinger [97, 98], Hitler [99], Harish-Chandra [100, 101, 102], Hoffmann [103], Utiyama [104], Gel'fand-Yaglom [105], Schouten [106], Gupta [107], Bleuler [108], Fujiwara [109], Borgardt [110, 111], Kuohsien [112], Hjalmar [113], Bogush-Fedorov [114], Beckers-Pirotte [115], Casanova [116], Krivski-Romamenko-Fushchych [117], Goldman-Tsai-Yildiz [118], Fushchych-Nikitin [119],

However, this technique till recent time was not used in the curved space-time when constructing explicit solutions, though that possibility was known - see Weinberg [144], Birrel-Davies [146]. The situation is changing now: Lunardi et al [120, 123, 124, 125], Fainberg-Pimentel [121], De Montigny et al [122], Red'kov [126, 127], Bogush et al [76, 128, 129]

In the present paper, this approach will be applied to the case of spin 1 particle in anti de Sitter space; previously, analogous treatment to a particle in de Sitter model was given in [76]. Then we will treat the same problem on the base of 5-dimensional wave equation. And in the end we will consider the problem within an approach, based on tetrad form of Maxwell equations in complex representation [166, 167], it seems be the most simple way to describe the classical electromagnetic field in curved space models.

2 Duffin – Kemmer – Petiau tetrad equation in Riemannian space-time

We start from a flat space equation in its matrix DKP-form

$$(i \beta^a \partial_a - \frac{mc}{\hbar}) \Phi(x) = 0 ; \quad (1)$$

where

$$\begin{aligned} \Phi &= (\Phi_0, \Phi_1, \Phi_2, \Phi_3; \Phi_{01}, \Phi_{02}, \Phi_{03}, \Phi_{23}, \Phi_{31}, \Phi_{12}) , \\ \beta^a &= \begin{vmatrix} 0 & \kappa^a \\ \lambda^a & 0 \end{vmatrix} = \kappa^a \oplus \lambda^a , \quad (\kappa^a)_j^{[mn]} = -i (\delta_j^m g^{na} - \delta_j^n g^{ma}) , \\ (\lambda^a)_j^{[mn]} &= -i (\delta_m^a \delta_n^j - \delta_n^a \delta_m^j) = -i \delta_{mn}^{aj} ; \end{aligned} \quad (2)$$

$(g^{na}) = \text{diag}(+1, -1, -1, -1)$. The basic properties of β^a are

$$\begin{aligned} \beta^c \beta^a \beta^b &= \begin{vmatrix} 0 & \kappa^c \lambda^a \kappa^b \\ \lambda^c \kappa^a \lambda^b & 0 \end{vmatrix} , \quad (\lambda^c \kappa^a \lambda^b)_j^{[mn]} = i (\delta_{mn}^{cb} g^{aj} - \delta_{mn}^{cj} g^{ab}) , \\ (\kappa^c \lambda^a \kappa^b)_j^{[mn]} &= i [\delta_j^a (g^{cm} g^{bn} - g^{cn} g^{bm}) + g^{ac} (\delta_j^n g^{mb} - \delta_j^m g^{nb})] , \end{aligned} \quad (3)$$

and

$$\begin{aligned} \beta^c \beta^a \beta^b + \beta^b \beta^a \beta^c &= \beta^c g^{ab} + \beta^b g^{ac} , \\ [\beta^c, j^{ab}] &= g^{ca} \beta^b - g^{cb} \beta^a , \quad j^{ab} = \beta^a \beta^b - \beta^b \beta^a , \\ [j^{mn}, j^{ab}] &= (g^{na} j^{mb} - g^{nb} j^{ma}) - (g^{ma} j^{nb} - g^{mb} j^{na}) . \end{aligned} \quad (4)$$

In accordance with tetrad recipe one should generalize the DKP-equation as follows [168]

$$\begin{aligned} [i \beta^\alpha(x) (\partial_\alpha + B_\alpha(x)) - \frac{mc}{\hbar}] \Phi(x) &= 0 , \\ \beta^\alpha(x) &= \beta^a e_{(a)}^\alpha(x) , \quad B_\alpha(x) = \frac{1}{2} j^{ab} e_{(a)}^\beta \nabla_\alpha (e_{(b)\beta}) . \end{aligned} \quad (5)$$

This equation contains the tetrad $e_{(a)}^\alpha(x)$ explicitly. Therefore, there must exist a possibility to demonstrate the equivalence of any variants of this equation associated with various tetrads:

$$e_{(a)}^\alpha(x) , \quad e_{(b)}'^\alpha(x) = L_a^b(x) e_{(b)}^\alpha(x) , \quad (6)$$

$L_a^b(x)$ is a local Lorentz transformation. We will show that two such equations

$$\begin{aligned} [i \beta^\alpha(x) (\partial_\alpha + B_\alpha(x)) - \frac{mc}{\hbar}] \Phi(x) &= 0 , \\ [i \beta'^\alpha(x) (\partial_\alpha + B'_\alpha(x)) - \frac{mc}{\hbar}] \Phi'(x) &= 0 , \end{aligned} \quad (7)$$

generated in tetrads $e_{(a)}^\alpha(x)$ and $e_{(b)}'^\alpha(x)$ respectively, can be converted into each other through a local gauge transformation:

$$\Phi'(x) = \begin{vmatrix} \phi'_a(x) \\ \phi'_{[ab]}(x) \end{vmatrix} = \begin{vmatrix} L_a^l & 0 \\ 0 & L_a^m L_b^n \end{vmatrix} \begin{vmatrix} \phi_l(x) \\ \phi_{[mn]}(x) \end{vmatrix} . \quad (8)$$

Indeed, tetrad DKP-equation manifests a gauge symmetry under local Lorentz transformations in complete analogy with more familiar Dirac particle case. In the same time, the wave function from this equation represents scalar quantity relative to general coordinate transformations: if $x^\alpha \rightarrow x'^\alpha = f^\alpha(x)$, then $\Phi'(x) = \Phi(x)$.

It remains to demonstrate that this *DKP* formulation can be inverted into the Proca formalism in terms of general relativity tensors. To this end, as a first step, let us allow for the sectional structure of β^a , J^{ab} and $\Phi(x)$ in the *DKP*-equation; then instead of (5) we get

$$\begin{aligned} i [\lambda^c e_{(c)}^\alpha (\partial_\alpha + \kappa^a \lambda^b e_{(a)}^\beta \nabla_\alpha e_{(b)\beta})]_{[mn]}^l \Phi_l &= \frac{mc}{\hbar} \Phi_{[mn]} , \\ i [\kappa^c e_{(c)}^\alpha (\partial_\alpha + \lambda^a \kappa^b e_{(a)}^\beta \nabla_\alpha e_{(b)\beta})]_l^{[mn]} \Phi_{[mn]} &= \frac{mc}{\hbar} \Phi_l , \end{aligned} \quad (9)$$

which lead to

$$\begin{aligned} (e_{(a)}^\alpha \partial_\alpha \Phi_b - e_{(b)}^\alpha \partial_\alpha \Phi_a) + (\gamma_{ab}^c - \gamma_{ba}^c) \Phi_c &= \frac{mc}{\hbar} \Phi_{ab} , \\ e^{(b)\alpha} \partial_\alpha \Phi_{ab} + \gamma^{nb}_n \Phi_{ab} + \gamma_a^{mn} \Phi_{mn} &= \frac{mc}{\hbar} \Phi_a ; \end{aligned} \quad (10)$$

the symbol $\gamma_{abc}(x)$ is used to designate Ricci coefficients:

$$\gamma_{abc}(x) = - e_{(a)\alpha;\beta} e_{(b)}^\alpha e_{(c)}^\beta .$$

In turn, (10) will look as the Proca equations

$$\nabla_\alpha \Psi_\beta - \nabla_\beta \Psi_\alpha = m \Psi_{\alpha\beta}, \quad \nabla^\beta \Psi_{\alpha\beta} = m \Psi_\alpha ; \quad (11)$$

they are rewritten in terms of tetrad components

$$\Phi_a = e_{(a)}^\alpha \Phi_\alpha , \quad \Phi_{ab} = e_{(a)}^\alpha e_{(b)}^\beta \Phi_{\alpha\beta} . \quad (12)$$

So, as evidenced by the above, the manner of introducing the interaction between a spin 1 particle and external classical gravitational field can be successfully unified with the approach that occurred with regard to a spin 1/2 particle and was first developed by Tetrode, Weyl, Fock, Ivanenko. One should attach great significance to that possibility of unification. Moreover, its absence would be a very strange fact. Let us add some more details.

The manner of extending the flat space Dirac equation to general relativity case indicates clearly that the Lorentz group underlies equally both these theories. In other words, the Lorentz group retains its importance and significance at changing the Minkowski space model to an arbitrary curved space-time. In contrast to this, at generalizing the Proca formulation, we automatically destroy any relations to the Lorentz group, although the definition itself for a spin 1 particle as an elementary object was based on this group. Such a gravity sensitiveness to the fermion-boson division might appear rather strange and unattractive asymmetry, being subjected to the criticism. Moreover, just this feature has brought about a plenty of speculations about this matter. In any case, this peculiarity of particle-gravity field interaction is recorded almost in every handbook on fields in curved space-times.

3 Separation of variables

In contrast to most of approaches used in the literature and based on group theoretical arguments we start our analysis of the spin 1 field with the use of the old and conventional tetrad formalism of Tetrode-Weyl-Fock-Ivanenko applied to matrix Duffin – Kemmer – Petiau formalism (see [168]). With the use of diagonal static spherical tetrad in anti de Sitter space-time $x^\alpha = (t, r, \theta, \phi)$ [145] and corresponding Ricci coefficients

$$\begin{aligned}
dS^2 &= (1 + r^2) dt^2 - \frac{dr^2}{1 + r^2} - r^2(d\theta^2 + \sin^2 \theta d\phi^2) , \\
\Phi &= 1 + r^2 , \quad g_{\alpha\beta} = \begin{vmatrix} \Phi & 0 & 0 & 0 \\ 0 & -1/\Phi & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{vmatrix} , \\
e_{(0)}^\alpha &= (\frac{1}{\sqrt{\Phi}}, 0, 0, 0) , \quad e_{(3)}^\alpha = (0, \sqrt{\Phi}, 0, 0) , \\
e_{(1)}^\alpha &= (0, 0, \frac{1}{r}, 0) , \quad e_{(2)}^\alpha = (1, 0, 0, \frac{1}{r \sin \theta}) , \\
\gamma_{030} &= \frac{r}{\sqrt{\Phi}} , \quad \gamma_{311} = \frac{\sqrt{\Phi}}{r} , \quad \gamma_{322} = \frac{\sqrt{\Phi}}{r} , \quad \gamma_{122} = \frac{\cos \theta}{r \sin \theta} ,
\end{aligned} \tag{13}$$

we get to explicit form of a matrix Duffin – Kemmer equation for a massive spin 1 particle

$$\begin{aligned}
&[i\beta^0 \partial_t + i\Phi(\beta^3 \partial_r + \frac{1}{r}(\beta^1 j^{31} + \beta^2 j^{32}) + \frac{\Phi'}{2\Phi} \beta^0 J^{03}) \\
&\quad + \frac{\sqrt{\Phi}}{r} \Sigma_{\theta, \phi} - m\sqrt{\Phi}] \Phi(x) = 0 , \\
&\Sigma_{\theta, \phi} = i \beta^1 \partial_\theta + \beta^2 \frac{i \partial + i j^{12} \cos \theta}{\sin \theta} .
\end{aligned} \tag{14}$$

Spherical waves with (j, m) quantum numbers should be constructed within the following general substitution (we adhere notation developed in Red'kov [139, 140, 141, 142]; before similar techniques was applied by Dray [135, 136], Krolkowski and Turski [137], Turski [138] ; many years ago such a tetrad basis was used by Schrödinger [130] and Pauli [131] when looking at the problem of single-valuedness of wave functions in quantum theory – then the case of spin $S = 1/2$ particle was specified; transition to spin 1 case is achieved in (14) trough a formal change of basic Dirac matrices into Duffin –Kemmer ones)

$$\begin{aligned}
\Phi_{\epsilon jm}(x) &= e^{-i\epsilon t} [f_1(r) D_0, f_2(r) D_{-1}, f_3(r) D_0, f_4(r) D_{+1}, \\
&f_5(r) D_{-1}, f_6(r) D_0, f_7(r) D_{+1}, f_8(r) D_{-1}, f_9(r) D_0, f_{10}(r) D_{+1}] ;
\end{aligned} \tag{15}$$

symbol D_σ designates Wigner [133] functions $D_{-m, \sigma}^j(\phi, \theta, 0)$ (we use notation according to the book [143],). In the literature equivalent techniques of spin-weighted harmonics Goldberg et al [134] (se also in [148]) preferably is used though equivalence of both approach is known [135, 136]. Requirement to diagonalize parity operator, $\hat{P} \Phi_{\epsilon jm} = P \Phi_{\epsilon jm}$, gives

$$P = (-1)^{j+1} , \quad f_1 = f_3 = f_6 = 0 , \quad f_4 = -f_2 ,$$

$$f_7 = -f_5, f_{10} = +f_8 ;$$

$$P = (-1)^j, \quad f_9 = 0, f_4 = +f_2, f_7 = +f_5, f_{10} = -f_8. \quad (16)$$

After separation of variables (for recursive relations needed see Section 10) we arrive to the radial systems ($\nu = \sqrt{j(j+1)}/2$):

$$P = (-1)^{j+1},$$

$$\begin{aligned} i\epsilon f_5 + i\Phi\left(\frac{d}{dr} + \frac{1}{r} + \frac{\Phi'}{2\Phi}\right) f_8 + i\nu\frac{\sqrt{\Phi}}{r} f_9 - m\sqrt{\Phi} f_2 &= 0, \\ i\epsilon f_2 - m\sqrt{\Phi} f_5 &= 0, \quad -i\Phi\left(\frac{d}{dr} + \frac{1}{r}\right) f_2 - m\sqrt{\Phi} f_8 = 0, \\ i2\nu\frac{\sqrt{\Phi}}{r} f_2 - m\sqrt{\Phi} f_9 &= 0; \end{aligned} \quad (17)$$

$$P = (-1)^j,$$

$$\begin{aligned} \Phi\left(\frac{d}{dr} + \frac{2}{r}\right) F_6 + \frac{2\nu}{r} F_5 + m F_1 &= 0, \\ i\epsilon F_5 + i\Phi\left(\frac{d}{dr} + \frac{1}{r}\right) F_8 - m\Phi F_2 &= 0, \\ i\epsilon F_6 - i2\nu r F_8 - m F_3 = 0, \quad -i\epsilon F_2 + \frac{\nu}{r} F_1 - m F_5 &= 0, \\ i\epsilon F_3 + \Phi\frac{d}{dr} F_1 + m\Phi F_6 &= 0, \\ i\Phi\left(\frac{d}{dr} + \frac{1}{r}\right) F_2 + i\frac{\nu}{r} F_3 + m F_8 &= 0; \end{aligned} \quad (18)$$

in (18) we have used substitutions

$$\begin{aligned} F_1 &= \sqrt{\Phi} f_1, F_2 = f_2, F_3 = \sqrt{\Phi} f_3, \\ F_5 &= \sqrt{\Phi} f_5, F_6 = f_6, F_8 = \sqrt{\Phi} f_8. \end{aligned}$$

The case of minimal value $j = 0$ is to be treated separately, because one must use a special substitution from the very beginning

$$\Phi_{\epsilon jm}(x) = e^{-i\epsilon t} (f_1, 0, f_3, 0, 0, f_6, 0, 0, f_9, 0); \quad (19)$$

The angular part of the wave operator $\Sigma_{\theta, \phi}$ acts as a zero operator and eq. (14) takes the form

$$\begin{aligned} [i\beta^0 \partial_t + i\Phi(\beta^3 \partial_r + \frac{1}{r}(\beta^1 j^{31} + \beta^2 j^{32}) \\ + \frac{\Phi'}{2\Phi} \beta^0 J^{03}) - m\sqrt{\Phi}] \Phi(x) = 0; \end{aligned} \quad (20)$$

correspondingly we have a very simple radial system

$$\begin{aligned} -\Phi\left(\frac{d}{dr} + \frac{2}{r}\right) f_6 - m\sqrt{\Phi} f_1 &= 0, \quad i\epsilon f_6 - m\sqrt{\Phi} f_3 = 0, \\ -i\epsilon f_3 - \Phi\left(\frac{d}{dr} + \frac{\Phi'}{2\Phi}\right) f_1 - m\sqrt{\Phi} f_6 &= 0, \quad f_9 = 0. \end{aligned} \quad (21)$$

System (21) describes states with parity $P = (-1)^0 = +1$; states with $P = (-1)^{0+1} = -1$ do not exist. The system (21) reduces to second order differential equation for f_6 :

$$\frac{d^2}{dr^2} f_6 + \frac{2(1+2r^2)}{r(1+r^2)} \frac{d}{dr} f_6 + \left[\frac{\epsilon^2}{(1+r^2)^2} - \frac{m^2-2}{1+r^2} - \frac{2}{r^2(1+r^2)} \right] f_6 = 0, \quad (22)$$

which is solved in hypergeometric functions

$$\begin{aligned} f_6(r) &= r (1+r^2)^{-\epsilon/2} F(\alpha, \beta, \gamma, -r^2), \\ \gamma &= 1 + 3/2, \quad \alpha = \frac{3/2 + 1 - \epsilon + \sqrt{m^2 + 1/4}}{2}, \\ \beta &= \frac{3/2 + 1 - \epsilon - \sqrt{m^2 + 1/4}}{2}. \end{aligned} \quad (23)$$

The hypergeometric series becomes a polynomial when $\alpha = -n$, $n = 0, 1, 2, \dots$; so we arrive at an energy quantization rule

$$\epsilon = N + 3/2 + \sqrt{m^2 + 1/4}, \quad N = 2n + 1 \in \{0, 1, 2, \dots\}. \quad (24)$$

4 Solutions of radial equations at $j > 0$

Let us turn to eqs. (17). Expressing f_5, f_8, f_9 through f_2

$$\begin{aligned} f_5 &= \frac{i}{m\sqrt{\Phi}} \epsilon f_2, \quad f_9 = \frac{i}{m} \frac{2\nu}{r} f_2, \\ f_8 &= -\frac{i}{m\sqrt{\Phi}} \Phi \left(\frac{d}{dr} + \frac{1}{r} \right) f_2, \end{aligned} \quad (25)$$

for f_2 we get

$$\frac{d^2}{dr^2} f_2 + \frac{2(1+2r^2)}{r(1+r^2)} \frac{d}{dr} f_2 + \left[\frac{\epsilon^2}{(1+r^2)^2} - \frac{m^2-2}{1+r^2} - \frac{j(j+1)}{r^2(1+r^2)} \right] f_2 = 0. \quad (26)$$

Below solutions of this type are referred as j -waves Eq. (26) is solved in hypergeometric functions

$$\begin{aligned} f_2 &= U_{\epsilon,j} = (-z)^{j/2} (1-z)^{-\epsilon/2} F(\alpha, \beta, \gamma; z), \quad \gamma = j + 3/2, \\ \alpha &= \frac{3/2 + j - \epsilon + \sqrt{m^2 + 1/4}}{2}, \quad \beta = \frac{3/2 + j - \epsilon - \sqrt{m^2 + 1/4}}{2}. \end{aligned} \quad (27)$$

Restriction $\alpha = -n$, $n = 0, 1, 2, \dots$ makes hypergeometric series polynomials, so we get a quantization rule for energy levels:

$$\epsilon = N + 3/2 + \sqrt{m^2 + 1/4}, \quad N = 2n + j \in \{0, 1, 2, \dots\}. \quad (28)$$

It is verified easily that as $z \rightarrow -\infty$ the radial function $U_{\epsilon,j}(z)$ tends to zero

$$U_{\epsilon,j}(z \rightarrow -\infty) \sim z^{j/2} z^{-\epsilon/2} z^n \sim z^{-3/4 - \sqrt{m^2+1/4}/2}.$$

Now let us turn to eqs. (18). With the use of two different substitutions

$$\begin{aligned} I. \quad F_1 &= \sqrt{j+1} G_1, \quad F_2 = i\sqrt{j/2} G_2, \quad F_3 = i\sqrt{j+1} G_3, \\ F_5 &= \sqrt{j/2} G_5, \quad F_6 = \sqrt{j+1} G_6, \quad F_8 = \sqrt{j/2} G_8; \end{aligned} \quad (29)$$

$$\begin{aligned} II. \quad F_1 &= \sqrt{j} G_1, \quad F_2 = i\sqrt{(j+1)/2} G_2, \quad F_3 = i\sqrt{j} G_3, \\ F_5 &= \sqrt{(j+1)/2} G_5, \quad F_6 = \sqrt{j} G_6, \quad F_8 = \sqrt{(j+1)/2} G_8, \end{aligned} \quad (30)$$

and expressing G_5, G_6, G_8 through G_1, G_2, G_3 we arrive at three equations respectively

I.

$$\begin{aligned} & \left(\frac{j(j+1)}{r^2} + m^2 - \Phi \left(\frac{d}{dr} + \frac{2}{r} \right) \frac{d}{dr} \right) G_1 \\ & + \frac{\epsilon j}{r} G_2 + \epsilon \Phi \left(\frac{d}{dr} + \frac{2}{r} \right) \frac{1}{\Phi} G_3 = 0, \\ & (\epsilon^2 - m^2 \Phi^2 + \Phi \left(\frac{d}{dr} + \frac{1}{r} \right) \Phi \left(\frac{d}{dr} + \frac{1}{r} \right)) G_2 \\ & + \frac{\epsilon(j+1)}{r} G_1 + \Phi \frac{j+1}{r} \frac{d}{dr} G_3 = 0, \\ & \left(\frac{\epsilon^2}{\Phi} - \left(\frac{j(j+1)}{r^2} m^2 \right) G_3 - \right. \\ & \left. - \epsilon \frac{d}{dr} G_1 - \frac{j}{r} \Phi \left(\frac{d}{dr} + \frac{1}{r} \right) G_2 = 0; \right. \end{aligned} \quad (31)$$

II.

$$\begin{aligned} & \left(\frac{j(j+1)}{r^2} + m^2 - \Phi \left(\frac{d}{dr} + \frac{2}{r} \right) \frac{d}{dr} \right) G_1 + \\ & + \frac{\epsilon(j+1)}{r} G_2 + \epsilon \Phi \left(\frac{d}{dr} + \frac{2}{r} \right) \frac{1}{\Phi} G_3 = 0, \\ & (\epsilon^2 - m^2 \Phi^2 + \Phi \left(\frac{d}{dr} + \frac{1}{r} \right) \Phi \left(\frac{d}{dr} + \frac{1}{r} \right)) G_2 \\ & + \frac{\epsilon j}{r} G_1 + \Phi \frac{j}{r} \frac{d}{dr} G_3 = 0, \\ & \left(\frac{\epsilon^2}{\Phi} - \frac{j(j+1)}{r^2} - m^2 \right) G_3 \\ & - \epsilon \frac{d}{dr} G_1 - \frac{(j+1)}{r} \Phi \left(\frac{d}{dr} + \frac{1}{r} \right) G_2 = 0. \end{aligned} \quad (32)$$

To solve eqs. (31) and (32), one can make use of the Lorentz condition. Its explicit form can easily be found

$$\frac{-i\epsilon}{\sqrt{\Phi}} f_1 - \sqrt{\Phi} \left(\frac{d}{dr} + \frac{2}{r} + \frac{\Phi'}{2\Phi} \right) f_3 - \frac{\nu}{r} (f_2 + f_4) = 0. \quad (33)$$

When $P = (-1)^{j+1}$ eq. (33) holds identically; for substitutions I and II in (29) and (30) it gives respectively

$$\begin{aligned} I. \quad & -\epsilon \frac{G_1}{\Phi} = \frac{j}{r} G_2 + \left(\frac{d}{dr} + \frac{2}{r} \right) G_3, \\ II. \quad & -\epsilon \frac{G_1}{\Phi} = \frac{j+1}{r} G_2 + \left(\frac{d}{dr} + \frac{2}{r} \right) G_3. \end{aligned} \quad (34)$$

Allowing for relations (34), let us express G_1 through G_2 and G_3 , and substitute the results into 2-nd and 3-d equations in (31) and (32). Thus we get respectively

I.

$$\begin{aligned} & \left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{\Phi'}{\Phi} \right) \frac{d}{dr} + \frac{\Phi'}{r\Phi} + \frac{\epsilon^2}{\Phi^2} \right. \\ & \left. - \frac{m^2}{\Phi} - \frac{j(j+1)}{\Phi r^2} \right] G_2 - \frac{2(j+1)}{r^2 \Phi} G_3 = 0, \\ & \left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{\Phi'}{\Phi} \right) \frac{d}{dr} + \frac{2\Phi'}{r\Phi} - \frac{2}{r^2} + \frac{\epsilon^2}{\Phi^2} \right. \\ & \left. - \frac{m^2}{\Phi} - \frac{j(j+1)}{\Phi r^2} \right] G_3 - \frac{2j}{r^2 \Phi} G_2 = 0; \end{aligned} \quad (35)$$

II.

$$\begin{aligned} & \left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{\Phi'}{\Phi} \right) \frac{d}{dr} + \frac{\Phi'}{r\Phi} + \frac{\epsilon^2}{\Phi^2} \right. \\ & \left. - \frac{m^2}{\Phi} - \frac{j(j+1)}{\Phi r^2} \right] G_2 - \frac{2j}{r^2 \Phi} G_3 = 0, \\ & \left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{\Phi'}{\Phi} \right) \frac{d}{dr} + \frac{2\Phi'}{r\Phi} - \frac{2}{r^2} + \frac{\epsilon^2}{\Phi^2} \right. \\ & \left. - \frac{m^2}{\Phi} - \frac{j(j+1)}{\Phi r^2} \right] G_3 - \frac{2(j+1)}{r^2 \Phi} G_2 = 0. \end{aligned} \quad (36)$$

In the case *I*, taking $G_3 = +G_2$, from two equations (35) we get one the same

$$\begin{aligned} I. \quad & G_3 = +G_2 = U_{\epsilon, j+1}, \quad \left[\frac{d^2}{dr^2} + \frac{2(1+2r^2)}{r(1+r^2)} \frac{d}{dr} \right. \\ & \left. + \frac{\epsilon^2}{(1+r^2)^2} - \frac{M^2-2}{1+r^2} - \frac{(j+1)(j+2)}{r^2(1+r^2)} \right] G_2 = 0. \end{aligned} \quad (37)$$

In the same manner, in the case *II*, taking $G_3 = -G_2$, we get one the same equation (it differs from previous one by the simple formal changing $(j+1)$ into $(j-1)$):

$$\begin{aligned} II. \quad & G_3 = -G_2 = U_{\epsilon, j-1} \quad \left[\frac{d^2}{dr^2} + \frac{2(1+2r^2)}{r(1+r^2)} \frac{d}{dr} \right. \\ & \left. + \frac{\epsilon^2}{(1+r^2)^2} - \frac{M^2-2}{1+r^2} - \frac{(j-1)j}{r^2(1+r^2)} \right] G_2 = 0. \end{aligned} \quad (38)$$

Thus, beside the waves of *j*-type, there exist else two types (all technical details of calculations with hypergeometric function are omitted)

I. $(j+1)$ - type ,

$$\begin{aligned}
G_3 = G_2 = U_{\epsilon,j+1} , \quad -\epsilon \frac{G_1}{\Phi} &= \left(\frac{d}{dr} + \frac{j+2}{r} \right) G_2 ; \\
G_1 &= \sqrt{-z} U_{\epsilon,j+1} - \frac{2j+3}{\epsilon} \sqrt{1-z} U_{\epsilon-1,j} , \\
U_{\epsilon,j+1} &= (-z)^{(j+1)/2} (1-z)^{-\epsilon/2} F(\alpha, \beta, \gamma; z) , \\
U_{\epsilon-1,j} &= (-z)^{j/2} (1-z)^{-(\epsilon-1)/2} F(\alpha, \beta, \gamma-1; z) , \\
\gamma = j+1+3/2 , \quad \alpha &= \frac{3/2+j+1-\epsilon+\sqrt{m^2+1/4}}{2} , \\
\beta &= \frac{3/2+j+1-\epsilon-\sqrt{m^2+1/4}}{2} ;
\end{aligned} \tag{39}$$

II. $(j-1)$ - type,

$$\begin{aligned}
-G_3 = G_2 = U_{\epsilon,j-1} , \quad -\epsilon \frac{G_1}{\Phi} &= \left(-\frac{d}{dr} + \frac{j-1}{r} \right) G_2 , \\
G_1 &= -\sqrt{-z} U_{\epsilon,j-1} - \frac{2}{\epsilon} \frac{\alpha\beta}{\gamma} \sqrt{1-z} U_{\epsilon-1,j} , \\
U_{\epsilon,j-1} &= (-z)^{(j-1)/2} (1-z)^{-\epsilon/2} F(\alpha, \beta, \gamma; z) , \\
U_{\epsilon-1,j} &= (-z)^{j/2} (1-z)^{-(\epsilon-1)/2} F(\alpha+1, \beta+1, \gamma+1; z) , \\
\gamma = j-1+3/2 , \quad \alpha &= \frac{3/2+j-1-\epsilon+\sqrt{m^2+1/4}}{2} , \\
\beta &= \frac{3/2+j-1-\epsilon-\sqrt{m^2+1/4}}{2} .
\end{aligned} \tag{40}$$

Let us collect results together. There are constructed solutions of three types (below only f_1, \dots, f_4 are specified):

j - wave

$$f_1 = f_3 = 0 , \quad f_2 = -f_4 = U_{\epsilon,j} ;$$

$(j+1)$ - wave,

$$\begin{aligned}
f_1 &= \sqrt{j+1} \left[\frac{\sqrt{-z}}{\sqrt{1-z}} U_{\epsilon,j+1} - \frac{2j+3}{\epsilon} U_{\epsilon-1,j} \right] , \\
f_2 = +f_4 &= +i \sqrt{j/2} U_{\epsilon,j+1} , \quad f_3 = +i \sqrt{j+1} \frac{1}{\sqrt{1-z}} U_{\epsilon,j+1} ;
\end{aligned}$$

$(j-1)$ - wave ,

$$\begin{aligned}
f_1 &= \sqrt{j} \left[-\frac{\sqrt{-z}}{\sqrt{1-z}} U_{\epsilon,j-1} - \frac{2}{\epsilon} \frac{\alpha\beta}{\gamma} U_{\epsilon-1,j} \right] , \\
f_2 = +f_4 &= i \sqrt{\frac{j+1}{2}} U_{\epsilon,j-1} , \quad f_3 = -i \sqrt{j} \frac{1}{\sqrt{1-z}} U_{\epsilon,j-1} .
\end{aligned} \tag{41}$$

Three types of solutions correspond to three possible values of the orbital angular moment for spin 1 particle at fixed j : $l = j, j+1, j-1$.

5 Massless limit for spin 1 particle

Let us shortly consider a massless limit. The Duffin – Kemmer – Petiau equation (14) stays much the same with only formal change

$$m \sqrt{\Phi} \rightarrow P_6 \sqrt{\Phi}, \quad P_6 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{vmatrix} \quad (42)$$

which produces evident alterations in the radial system

$$\begin{aligned} m\sqrt{\Phi}f_i &\rightarrow 0, \quad \text{at } i = 1, 2, 3, 4; \\ m\sqrt{\Phi}f_i &\rightarrow \sqrt{\Phi}f_i, \quad \text{at } i = 5, \dots, 10. \end{aligned} \quad (43)$$

In massless case the Lorentz condition must be considered as an external gauge restriction for photon field. Other relations remain the same, instead of old parameters of hypergeometric functions now one must take new ones

$$U_{\epsilon,j}, \quad \alpha = \frac{2+j-\epsilon}{2}, \quad \beta = \frac{1+j-\epsilon}{2}, \quad \gamma = j+3/2; \quad (44)$$

the energy quantization rule looks

$$\epsilon = 2n + j + 2 = N + 2, \quad N = 2n + j \in \{0, 1, 2, \dots\}. \quad (45)$$

The case of minimal value $j = 0$ should be considered separately. The system (21) becomes

$$\begin{aligned} -\Phi \left(\frac{d}{dr} + \frac{2}{r} \right) f_6 - 0 \sqrt{\Phi} f_1 &= 0, \quad i\epsilon f_6 - 0 \sqrt{\Phi} f_3 = 0, \\ -i\epsilon f_3 - \Phi \left(\frac{d}{dr} + \frac{\Phi'}{2\Phi} \right) f_1 - \sqrt{\Phi} f_6 &= 0, \quad f_9 = 0, \end{aligned} \quad (46)$$

which is equivalent to

$$g_6 = 0, f_9 = 0, \quad -i\epsilon f_3 - \Phi \left(\frac{d}{dr} + \frac{\Phi'}{2\Phi} \right) f_1 = 0, \quad (47)$$

Therefore, for all states of electromagnetic field at $j = 0$ the components of electric and magnetic vectors vanish ($F_{\alpha\beta} = 0$); and non-vanishing $f_1(r), f_3(r)$ correspond to solutions of gradient type $A_\alpha = \nabla_\alpha \Phi$. To have fixed two radial functions in (47), one must impose certain gauge condition. In particular, taking the Lorentz condition (see (33)), we get equations (let it be $f_1 = \Phi^{-1/2} F_1, f_3 = \Phi^{-1/2} F_3$):

$$-\frac{i\epsilon}{\Phi} F_3 - \frac{d}{dr} F_1 = 0, \quad -\frac{i\epsilon}{\Phi} F_1 - \left(\frac{d}{dr} + \frac{2}{r} \right) F_3 = 0; \quad (48)$$

from whence it follows

$$\left[\frac{d^2}{dr^2} + \frac{2(1+2r^2)}{r(1+r^2)} \frac{d}{dr} + \frac{\epsilon^2}{(1+r^2)^2} \right] F_1 = 0, \quad (49)$$

which represent $j = 0$ -spherical solution of the equation $\nabla^\alpha \nabla_\alpha \Phi = 0, \Phi = e^{-i\epsilon t} f(r)$.

6 5-dimensional form of the wave equation

It is well known that wave equation for a particle with spin 1 in the de Sitter and anti de Sitter spaces can be presented in 5-dimensional form explicitly invariant under groups $SO(4,1)$ and $SO(3,2)$ respectively. Let us specify the problem of spherical wave solutions in anti de Sitter model to the 5-dimensional formalism. It is convenient to start with conformal flat coordinates

$$dS^2 = \frac{(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2}{\Phi^2}, \quad \Phi = (1 + x^2)/2; \quad (50)$$

the Proca tensor equations read

$$\partial_\alpha \Psi_\beta - \partial_\beta \Psi_\alpha = m \Psi_{\alpha\beta}, \quad \Phi^2 \partial^\beta \Psi_{\alpha\beta} = m \Psi_\alpha. \quad (51)$$

Let us introduce five coordinates ξ^a ($a = \alpha, 5$):

$$\begin{aligned} \xi^\alpha &= \frac{x^\alpha}{\Phi}, & \xi^5 &= \frac{1 - x^2}{1 + x^2}, \\ x^\alpha &= \frac{\xi^\alpha}{1 + \xi^5}, & \Phi &= \frac{1}{1 + \xi^5}, \\ (\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 + (\xi^5)^2 &= +1, \\ dS^2 &= \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta + (d\xi^5)^2. \end{aligned} \quad (52)$$

In other words, the anti de Sitter space-time can be considered as a sphere in 5-dimensional pseudo-Euclidean space; therefore the model permits 10-parametric symmetry group $SO(3,2)$. Instead $\Psi^\alpha(x)$ (in the following designated as $a^\alpha(x)$) let us introduce 5-vector $A^a(\xi)$:

$$\begin{aligned} A^\alpha &= \left(\frac{\delta^\alpha_\beta}{\Phi} - \frac{x^\alpha x_\beta}{\Phi^2} \right) a^\beta, & A^5 &= -\frac{x_\beta a^\beta}{\Phi^2}, \\ a^\alpha(x) &= \Phi (A^\alpha - x^\alpha A^5). \end{aligned} \quad (53)$$

These five components $A^a(\xi)$ obey additional restriction

$$A^a \xi_a = A^0 \xi^0 - \vec{A} \cdot \vec{\xi} + A^5 \xi^5 = 0.$$

Invariant with respect to $SO(3,2)$ wave equation for vector $A^a(\xi)$ should be constructed with the help of the operator $L_{ab} = \xi_a (\partial/\partial \xi^b) - \xi_b (\partial/\partial \xi^a)$ and looks as follows

$$\begin{aligned} \left(-\frac{1}{2} L^{ab} L_{ab} + m^2 - 2 \right) A_c &= 0, \\ L_{ab} A^b &= A_a, & A^a \xi_a &= 0. \end{aligned} \quad (54)$$

7 Spherical waves in 5-dimensional form, separation of variables

Let us consider equations (54) in static coordinates of the anti de Sitter space

$$\xi^1 = r \sin \theta \cos \phi, \quad \xi^2 = r \sin \theta \sin \phi, \quad \xi^3 = r \cos \theta,$$

$$\begin{aligned}
\xi^0 &= \sin t \sqrt{1+r^2} , \quad \xi^5 = \cos t \sqrt{1+r^2} ; \\
t &= \operatorname{arctg} \frac{\xi^0}{\xi^5} , \quad r = \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2} , \\
\theta &= \operatorname{arctg} \frac{\sqrt{(\xi^1)^2 + (\xi^2)^2}}{\xi^3} , \quad \phi = \operatorname{arctg} \frac{\xi^2}{\xi^1} .
\end{aligned} \tag{55}$$

For any representation of the group $SO(3,2)$ on the functions $\Psi(\xi)$ we have

$$\xi' = S \xi , \quad \Psi'(\xi') = U \Psi(\xi) \quad \implies \quad \Psi'(\xi) = U \Psi(S^{-1} \xi) .$$

In particular case $U \equiv S$ and $\Psi \equiv A$, for rotation in the plane 0 – 5

$$\xi^{0'} = \cos \omega \xi^0 - \sin \omega \xi^5 , \quad \xi^{5'} = \sin \omega \xi^0 + \cos \omega \xi^5 ,$$

we get

$$\begin{aligned}
A'(\xi) &= (I + \delta \omega J_{50}) A(\xi) , \quad J_{50} = \mathcal{L}_{50} + \sigma_{50} , \\
\mathcal{L}_{50} &= \xi_5 \frac{\partial}{\partial \xi^0} - \xi^0 \frac{\partial}{\partial \xi^5} , \quad \sigma_{50} = \begin{vmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 \end{vmatrix} .
\end{aligned} \tag{56}$$

Generators will be used to construct 5-form for energy operator, and total quantum moment. From requirements

$$\begin{aligned}
(+iJ_{50})^a_b A^b &= \epsilon A^a , \\
(\vec{J}^2)^a_b A^b &= j(j+1) A^a , \quad (J_3)^a_b A^b = m A^a
\end{aligned} \tag{57}$$

it follows

$$\begin{aligned}
\vec{A} &= e^{-i\epsilon t} \left[f(r) \vec{Y}_{jm}^{j+1}(\theta, \phi) + g(r) \vec{Y}_{jm}^{j-1}(\theta, \phi) + h(r) \vec{Y}_{jm}^j(\theta, \phi) \right] , \\
A^0 &= \left[e^{-i(\epsilon-1)t} F(r) + i e^{-i(\epsilon+1)t} G(r) \right] Y_{jm}(\theta, \phi) , \\
A^5 &= \left[i e^{-i(\epsilon-1)t} F(r) + e^{-i(\epsilon+1)t} G(r) \right] Y_{jm}(\theta, \phi) .
\end{aligned} \tag{58}$$

At given $j = 1, 2, \dots$ there exist three linearly independent spherical vectors $\nu = j+1, j, j-1$, when $j = 0$ there exist only one that

$$\begin{aligned}
j &= 0 , \quad \vec{A} = e^{-i\epsilon t} f(r) \vec{Y}_{00}^1 , \\
\frac{1}{2}(A^0 + iA^5) &= iG(r) e^{-i(\epsilon+1)t} , \\
\frac{1}{2}(A^0 - iA^5) &= F(r) e^{-i(\epsilon-1)t} .
\end{aligned} \tag{59}$$

Radial functions $f(r)$, $g(r)$, $h(r)$, $F(r)$, $G(r)$ should be determined from (54). From the first equation in (54), taking into account action of \vec{l}^2 on spherical vectors [143]

$$\begin{aligned}
\vec{l}^2 \vec{Y}_{jm}^\nu &= \nu(\nu+1) \vec{Y}_{jm}^\nu , \\
\vec{l}^2 Y_{jm} &= j(j+1) Y_{jm} , \quad \nu = j, j+1, j-1 ,
\end{aligned}$$

for radial functions $f(r)$, $g(r)$, $h(r)$, $F(r)$ we get equations of one the same type

$$\left[\frac{d^2}{dr^2} + \frac{2(1+2r^2)}{r(1+r^2)} \frac{d}{dr} + \frac{\Lambda^2}{(1+r^2)^2} - \frac{\nu(\nu+1)}{r^2(1+r^2)} - \frac{(m^2-2)}{1+r^2} \right] U_{\Lambda,\nu} = 0 ; \quad (60)$$

so we know these five functions to within numerical constants f_0, g_0, h_0, F_0, G_0

$$f = f_0 U_{\epsilon,j+1} , \quad g = g_0 U_{\epsilon,j-1} , \quad h = h_0 U_{\epsilon,j} , \\ F = F_0 U_{\epsilon-1,j} , \quad G = G_0 U_{\epsilon+1,j} ;$$

these constants should be obtained from remaining equations in (54). Solutions of eq. (60) are constructed in terms of hypergeometric functions (it suffices to consider in detail only the case $U_{\epsilon,j}$):

$$U_{\epsilon,j} = (-z)^{j/2} (1-z)^{-\epsilon/2} F(\alpha, \beta, \gamma; z) , \quad z = -r^2, \quad \gamma = j + 3/2 , \\ \alpha = \frac{3/2 + j - \epsilon + \sqrt{m^2 + 1/4}}{2} , \quad \beta = \frac{3/2 + j - \epsilon - \sqrt{m^2 + 1/4}}{2} ; \quad (61)$$

we have polynomials when $\alpha = -n$, $n = 0, 1, 2, \dots$; which results in the quantization rule for energy levels

$$\epsilon = N + 3/2 + \sqrt{m^2 + 1/4} , \quad N = 2n + j \in \{0, 1, 2, \dots\} . \quad (62)$$

From two remaining equations in (54) one can produce relationships between $(G \pm iF)$ and (f, g) :

$$G - iF = \frac{1}{\epsilon} \sqrt{1+r^2} \left[\left(\frac{d}{dr} + \frac{j+2}{r} \right) f - \left(\frac{d}{dr} - \frac{j-1}{r} \right) g \right] , \\ G + iF = \frac{-rf + rg}{\sqrt{1+r^2}} . \quad (63)$$

8 Solutions of the types $(j, j+1, j-1)$

Let us search three linearly independent solutions in the form

$$j - \text{type} , \quad f = 0 , \quad g = 0 , \quad h \neq 0 , \\ (j+1) - \text{type} , \quad f \neq 0 , \quad g = 0 , \quad h = 0 , \\ (j-1) - \text{type} , \quad f = 0 , \quad g \neq 0 , \quad h = 0 .$$

In fact, these requirements are equivalent to diagonalizing of the orbital angular operator $\vec{l}^2 = \nu(\nu+1)$, $\nu = j+1, j, j-1$. First, let us consider the wave $(j+1)$

$$f = \sqrt{\frac{2j+1}{j+1}} f_0 U_{\epsilon,j+1} , \quad F = F_0 U_{\epsilon-1,j} , \quad G = G_0 U_{\epsilon+1,j} ; \quad (64)$$

eqs. (63) take the form

$$G + iF = -\frac{rf(r)}{\sqrt{1+r^2}}, \quad G - iF = \frac{1}{\epsilon} \sqrt{1+r^2} \left(\frac{d}{dr} + \frac{j+2}{r} \right) f. \quad (65)$$

or after transition to the variable $z = -r^2$

$$\begin{aligned} 2\frac{G_0}{f_0} U_{\epsilon+1,j} &= -\frac{\sqrt{-z}}{\sqrt{1-z}} U_{\epsilon,j+1} + \frac{1}{\epsilon} \sqrt{1-z} \left(-2\sqrt{-z} \frac{d}{dz} + \frac{j+2}{\sqrt{-z}} \right) U_{\epsilon,j+1}, \\ 2i\frac{F_0}{f_0} U_{\epsilon-1,j} &= -\frac{\sqrt{-z}}{\sqrt{1-z}} U_{\epsilon,j+1} - \frac{1}{\epsilon} \sqrt{1-z} \left(-2\sqrt{-z} \frac{d}{dz} + \frac{j+2}{\sqrt{-z}} \right) U_{\epsilon,j+1}. \end{aligned} \quad (66)$$

Allowing for explicit forms

$$\begin{aligned} U_{\epsilon,j+1} &= (-z)^{(j+1)/2} (1-z)^{-\epsilon/2} F(\alpha, \beta, \gamma; z), \\ U_{\epsilon+1,j} &= (-z)^{j/2} (1-z)^{-(\epsilon+1)/2} F(\alpha-1, \beta-1, \gamma-1; z), \\ U_{\epsilon-1,j} &= (-z)^{j/2} (1-z)^{-(\epsilon-1)/2} F(\alpha, \beta, \gamma-1; z), \\ \gamma &= j+1+3/2, \quad \alpha = \frac{3/2+j+1-\epsilon+\sqrt{m^2+1/4}}{2}, \\ \beta &= \frac{3/2+j+1-\epsilon-\sqrt{m^2+1/4}}{2}, \end{aligned} \quad (67)$$

and using known formulas for hypergeometric functions we get expressions for G_0, F_0 :

$$G_0 = \frac{\gamma-1}{\epsilon} f_0 = \frac{j+3/2}{\epsilon} f_0, \quad F_0 = i \frac{j+3/2}{\epsilon} f_0 = i \frac{1-\gamma}{\alpha+\beta-\gamma} f_0. \quad (68)$$

Analogous calculations may be performed for the case of $(j-1)$ -waves:

$$\begin{aligned} g &= \sqrt{\frac{2j+1}{j}} g_0 U_{\epsilon,j-1}(z), \quad F = F_0 U_{\epsilon-1,j}, \quad G = G_0 U_{\epsilon+1,j}, \\ G + iF &= \frac{rg}{\sqrt{1+r^2}}, \quad G - iF = -\frac{1}{\epsilon} \sqrt{1+r^2} \left(\frac{d}{dr} - \frac{j-1}{r} \right) g; \end{aligned} \quad (69)$$

or in variable $z = -r^2$

$$\begin{aligned} 2\frac{G_0}{g_0} U_{\epsilon+1,j} &= \frac{\sqrt{-z}}{\sqrt{1-z}} U_{\epsilon,j-1} - \frac{1}{\epsilon} \sqrt{1-z} \left(-2\sqrt{-z} \frac{d}{dz} - \frac{j-1}{\sqrt{-z}} \right) U_{\epsilon,j-1}, \\ 2i\frac{F_0}{g_0} U_{\epsilon-1,j} &= \frac{\sqrt{-z}}{\sqrt{1-z}} U_{\epsilon,j-1} + \frac{1}{\epsilon} \sqrt{1-z} \left(-2\sqrt{-z} \frac{d}{dz} - \frac{j-1}{\sqrt{-z}} \right) U_{\epsilon,j-1}. \end{aligned} \quad (70)$$

Using the formulas

$$\begin{aligned} U_{\epsilon,j-1} &= (-z)^{(j-1)/2} (1-z)^{-\epsilon/2} F(a, b, c; z), \\ U_{\epsilon+1,j} &= (-z)^{j/2} (1-z)^{-(\epsilon+1)/2} F(a, b, c+1; z), \end{aligned}$$

$$\begin{aligned}
U_{\epsilon-1,j} &= (-z)^{j/2} (1-z)^{-(\epsilon-1)/2} F(a+1, b+1, c+1; z), \\
c &= j-1+3/2, \quad a = \frac{3/2+j-1-\epsilon+\sqrt{m^2+1/4}}{2}, \\
b &= \frac{3/2+j-1-\epsilon-\sqrt{m^2+1/4}}{2}
\end{aligned}$$

we arrive at

$$G_0 = \frac{(a-c)(b-c)}{\epsilon c} g_0, \quad F_0 = i \frac{ab}{\epsilon c} g_0. \quad (71)$$

Collecting together results we have

j -wave $j = 1, 2, 3, \dots$

$$\vec{A} = e^{-i\epsilon t} h_0 U_{-i\epsilon,j}(r) \vec{Y}_{jm}^j(\theta, \phi), \quad A^0 = 0, \quad A^5 = 0; \quad (72)$$

quantization rule $\epsilon = 2n + j + 3/2 + \sqrt{m^2 + 1/4}$.

$(j-1)$ -wave $j = 1, 2, 3, \dots$

$$\begin{aligned}
\vec{A} &= e^{-i\epsilon t} \sqrt{\frac{2j+1}{j}} f(r) \vec{J}_{jm}^{j-1}(\theta, \phi), \quad f(r) = f_0 U_{\epsilon,j-1}, \\
\frac{1}{2}(A^0 + iA^5) &= i G(r) e^{-i(\epsilon+1)t} Y_{jm}, \quad \frac{1}{2}(A^0 - iA^5) = F(r) e^{-i(\epsilon-1)t} Y_{jm}, \\
G(r) &= \frac{(a-c)(b-c)}{\epsilon c} g_0 U_{\epsilon+1,j}, \quad F(r) = i \frac{ab}{\epsilon c} g_0 U_{\epsilon-1,j},
\end{aligned} \quad (73)$$

quantization rule $\epsilon = 2n + j - 1 + 3/2 + \sqrt{m^2 + 1/4}$.

$(j+1)$ -wave, $j = 0, 1, 2, 3, \dots$

$$\begin{aligned}
\vec{A} &= e^{-i\epsilon t} \sqrt{\frac{2j+1}{j+1}} f(r) \vec{J}_{jm}^{j+1}(\theta, \phi), \quad f(r) = f_0 U_{\epsilon,j+1}, \\
\frac{1}{2}(A^0 + iA^5) &= i G(r) e^{-i(\epsilon+1)t} Y_{jm}, \quad G(r) = \frac{j+3/2}{\epsilon} f_0 U_{\epsilon+1,j}, \\
\frac{1}{2}(A^0 - iA^5) &= F(r) e^{-i(\epsilon-1)t} Y_{jm}, \quad F(r) = i \frac{j+3/2}{\epsilon} f_0 U_{\epsilon-1,j},
\end{aligned} \quad (74)$$

quantization rule $\epsilon = 2n + j + 1 + 3/2 + \sqrt{m^2 + 1/4}$. Degeneration of the energy levels can be

clarified by the table

	j – type	$(j - 1)$ – type	$(j + 1)$ – type
$N = 1$	$n = 0, j = 1$	$n = 0, j = 2$	$n = 0, j = 0$
$N = 2$	$n = 0, j = 2$	$n = 0, j = 3$	$n = 0, j = 1$
		$n = 0, j = 1$	
$N = 3$	$n = 0, j = 3$	$n = 0, j = 4$	$n = 0, j = 2$
	$n = 1, j = 1$	$n = 1, j = 2$	$n = 1, j = 0$
$N = 4$	$n = 0, j = 4$	$n = 0, j = 5$	$n = 0, j = 3$
	$n = 1, j = 2$	$n = 1, j = 3$	$n = 1, j = 1$
$N = 5$	$n = 0, j = 5$	$n = 0, j = 6$	$n = 0, j = 4$
	$n = 1, j = 3$	$n = 1, j = 4$	$n = 1, j = 2$
	$n = 2, j = 1$	$n = 2, j = 2$	$n = 2, j = 0$
$N = 6$	$n = 0, j = 6$	$n = 0, j = 7$	$n = 0, j = 5$
	$n = 1, j = 4$	$n = 1, j = 5$	$n = 1, j = 3$
	$n = 2, j = 2$	$n = 2, j = 3$	$n = 2, j = 1$
		$n = 3, j = 1$	
$N = 7$	$n = 0, j = 7$	$n = 0, j = 8$	$n = 0, j = 6$
	$n = 1, j = 5$	$n = 1, j = 6$	$n = 1, j = 4$
	$n = 2, j = 3$	$n = 2, j = 4$	$n = 2, j = 2$
	$n = 3, j = 1$	$n = 3, j = 2$	$n = 3, j = 0$
$N = 8$	$n = 0, j = 8$	$n = 0, j = 9$	$n = 0, j = 7$
	$n = 1, j = 6$	$n = 1, j = 7$	$n = 1, j = 5$
	$n = 2, j = 4$	$n = 2, j = 5$	$n = 2, j = 3$
	$n = 3, j = 2$	$n = 3, j = 3$	$n = 3, j = 1$
		$n = 4, j = 1$	

where energy level are given by(at $j = 0$, we have $\nu = j + 1 = 1$)

$$\epsilon = N + \frac{3}{2} + \sqrt{m^2 + 1/4}, \quad N = 2n + \nu, \quad j = 1, 2, 3, \dots, \quad \nu = j, j - 1, j + 1. \quad (75)$$

9 Tetrad form of Maxwell equations in Riemann – Silberstein – Majorana – Oppenheimer approach

It is well-known that Special Relativity arose from investigation of symmetry properties of the Maxwell equations with respect to inertial motion of the reference frame: Lorentz [149], Poincaré [150], Einstein [151]. Naturally, it was electromagnetic field that was the first and principal object for the Special Relativity: Minkowski [152], Silberstein [153], Marcolongo [155], Bateman [156]. In 1931 Majorana [158] and Oppenheimer [157] proposed to consider classical Maxwell equations as a quantum photon equations. In this context they introduced 3-vector function obeying Dirac-like massless wave equation. It turned out that much earlier in 1907 the same mathematical translation of classical Maxwell theory was performed by Silberstein

[153]; besides, he noted himself that the same approach was used earlier by Riemann [154]. That history was much forgotten, and many years this complex approach to electrodynamics was connected mainly with Majorana and Oppenheimer. Historical justice was rendered by Bialynicki-Birula [159], see also in Sipe [160], Gersten [161], Esposito [162], Dvoeglazov [163], Ivezić [164], Varlamov [165], Red'kov et al [166].

Below we use the complex formalism of Riemann – Silberstein – Majorana – Oppenheimer in Maxwell electrodynamics extended to the case of arbitrary pseudo-Riemannian space – time in accordance with the tetrad recipe of Tetrad – Weyl – Fock – Ivanenko (for more detail, see [167, 168]).

Maxwell equations in Riemann space can be presented in Riemann – Silberstein – Majorana – Oppenheimer basis as one matrix equation

$$\alpha^c (e_{(c)}^\rho \partial_\rho + \frac{1}{2} j^{ab} \gamma_{abc}) \Psi = J(x) ,$$

$$\alpha^0 = -iI , \quad \Psi = \begin{vmatrix} 0 \\ \mathbf{E} + ic\mathbf{B} \end{vmatrix} , \quad J = \frac{1}{\epsilon_0} \begin{vmatrix} \rho \\ i\mathbf{j} \end{vmatrix} \quad (76)$$

or

$$-i(e_{(0)}^\rho \partial_\rho + \frac{1}{2} j^{ab} \gamma_{ab0}) \Psi + \alpha^k (e_{(k)}^\rho \partial_\rho + \frac{1}{2} j^{ab} \gamma_{abk}) \Psi = J(x) . \quad (77)$$

Allowing for identities

$$\frac{1}{2} j^{ab} \gamma_{ab0} = [s_1(\gamma_{230} + i\gamma_{010}) + s_2(\gamma_{310} + i\gamma_{020}) + s_3(\gamma_{120} + i\gamma_{030})] ,$$

$$\frac{1}{2} j^{ab} \gamma_{abk} = [s_1(\gamma_{23k} + i\gamma_{01k}) + s_2(\gamma_{31k} + i\gamma_{02k}) + s_3(\gamma_{12k} + i\gamma_{03k})] ,$$

and using the notation

$$e_{(0)}^\rho \partial_\rho = \partial_{(0)} , \quad e_{(k)}^\rho \partial_\rho = \partial_{(k)} , \quad a = 0, 1, 2, 3 ,$$

$$(\gamma_{01a}, \gamma_{02a}, \gamma_{03a}) = \mathbf{v}_a , \quad (\gamma_{23a}, \gamma_{31a}, \gamma_{12a}) = \mathbf{p}_a , \quad (78)$$

eq. (77) in absence of sources reduces to

$$-i[\partial_{(0)} + \mathbf{s}(\mathbf{p}_0 + i\mathbf{v}_0)] \Psi + \alpha^k [\partial_{(k)} + \mathbf{s}(\mathbf{p}_k + i\mathbf{v}_k)] \Psi = 0 , \quad (79)$$

where

$$s_1 = \begin{vmatrix} 0 & 0 \\ 0 & \tau_1 \end{vmatrix} , \quad s_2 = \begin{vmatrix} 0 & 0 \\ 0 & \tau_1 \end{vmatrix} , \quad s_3 = \begin{vmatrix} 0 & 0 \\ 0 & \tau_1 \end{vmatrix} ,$$

$$\tau_1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} , \quad \tau_2 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix} , \quad \tau_3 = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} . \quad (80)$$

With the use of spherical tetrad in the anti de Sitter space the main equation (79) takes the form

$$\left[-\frac{i\partial_t}{\sqrt{\Phi}} + \sqrt{\Phi}(\alpha^3 \partial_r + \frac{\alpha^1 s_2 - \alpha^2 s_1}{r} + \frac{r}{\Phi} s_3) + \frac{1}{r} \Sigma_{\theta, \phi} \right] \begin{vmatrix} 0 \\ \psi \end{vmatrix} = 0 ,$$

$$\Sigma_{\theta, \phi} = \frac{\alpha^1}{r} \partial_\theta + \alpha^2 \frac{\partial_\phi + s_3 \cos \theta}{\sin \theta} . \quad (81)$$

It is convenient to have the spin matrix s_3 as diagonal, which is reached by simple linear transformation to the known cyclic basis

$$\begin{aligned} \Psi' &= U_4 \Psi, \quad U_4 = \begin{vmatrix} 1 & 0 \\ 0 & U \end{vmatrix}, \\ U &= \begin{vmatrix} -1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \end{vmatrix}, \quad U^{-1} = \begin{vmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{vmatrix}. \end{aligned} \quad (82)$$

so that

$$\begin{aligned} U\tau_1 U^{-1} &= \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -i & 0 \\ -i & 0 & -i \\ 0 & -i & 0 \end{vmatrix} = \tau'_1, \quad j'^{23} = s'_1 = \begin{vmatrix} 0 & 0 \\ 0 & \tau'_1 \end{vmatrix}, \\ U\tau_2 U^{-1} &= \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \tau'_2, \quad j'^{31} = s'_2 = \begin{vmatrix} 0 & 0 \\ 0 & \tau'_2 \end{vmatrix}, \end{aligned}$$

$$\begin{aligned} U\tau_3 U^{-1} &= -i \begin{vmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \tau'_3, \quad j'^{12} = s'_3 = \begin{vmatrix} 0 & 0 \\ 0 & \tau'_3 \end{vmatrix}. \\ \alpha'^1 &= \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -i & 0 \\ 0 & -i & 0 & -i \\ -1 & 0 & -i & 0 \end{vmatrix}, \quad \alpha'^2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -i & 0 & -i \\ -i & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -i & 0 & 1 & 0 \end{vmatrix}, \quad \alpha'^3 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +i \end{vmatrix}. \end{aligned}$$

Eq. (81) becomes

$$\begin{aligned} \left[-\frac{i\partial_t}{\sqrt{\Phi}} + \sqrt{\Phi}(\alpha'^3 \partial_r + \frac{\alpha'^1 s'_2 - \alpha'^2 s'_1}{r} + \frac{r}{\Phi} s'_3) + \frac{1}{r} \Sigma'_{\theta, \phi} \right] \begin{vmatrix} 0 \\ \psi' \end{vmatrix} &= 0, \\ \Sigma'_{\theta, \phi} &= \frac{\alpha'^1}{r} \partial_\theta + \alpha'^2 \frac{\partial_\phi + s'_3 \cos \theta}{\sin \theta}. \end{aligned} \quad (83)$$

10 Separating the variables and Wigner functions

Let us diagonalize operators \mathbf{J}^2, J^3 – corresponding substitution for ψ is

$$\psi = e^{-i\omega t} \begin{vmatrix} 0 \\ f_1(r) D_{-1} \\ f_2(r) D_0 \\ f_3(r) D_{+1} \end{vmatrix} \quad (84)$$

where the shorted notation for Wigner D -functions $D_\sigma = D_{-m, \sigma}^j(\phi, \theta, 0)$, $\sigma = -1, 0, +1$; j, m determine total angular momentum. With the use the following recursive relations [143]

$$\partial_\theta D_{-1} = \frac{1}{2}(a D_{-2} - \nu D_0), \quad \frac{m - \cos \theta}{\sin \theta} D_{-1} = \frac{1}{2}(a D_{-2} + \nu D_0),$$

$$\begin{aligned}
\partial_\theta D_0 &= \frac{1}{2}(\nu D_{-1} - \nu D_{+1}), \quad \frac{m}{\sin \theta} D_0 = \frac{1}{2}(\nu D_{-1} + \nu D_{+1}), \\
\partial_\theta D_{+1} &= \frac{1}{2}(\nu D_0 - a D_{+2}), \quad \frac{m + \cos \theta}{\sin \theta} D_{+1} = \frac{1}{2}(\nu D_0 + a D_{+2}), \\
\nu &= \sqrt{j(j+1)}, \quad a = \sqrt{(j-1)(j+2)}.
\end{aligned} \tag{85}$$

we get (the factor $e^{-i\omega t}$ is omitted)

$$\Sigma'_{\theta\phi} \Psi' = \frac{\nu}{\sqrt{2}} \begin{vmatrix} (f_1 + f_3)D_0 \\ -i f_2 D_{-1} \\ i(f_1 - f_3)D_0 \\ +i f_2 D_{+1} \end{vmatrix} \tag{86}$$

Turning back to Maxwell equation (83), after simple calculation we arrive at the radial system

$$\begin{aligned}
(1) \quad & \sqrt{\Phi} \left(\frac{d}{dr} + \frac{2}{r} \right) f_2 + \frac{1}{r} \frac{\nu}{\sqrt{2}} (f_1 + f_3) = 0, \\
(2) \quad & \left(-\frac{\omega}{\sqrt{\Phi}} - i \sqrt{\Phi} \frac{d}{dr} - i \frac{\sqrt{\Phi}}{r} - i \frac{r}{\sqrt{\Phi}} \right) f_1 - \frac{i}{r} \frac{\nu}{\sqrt{2}} f_2 = 0, \\
(3) \quad & -\frac{\omega}{\sqrt{\Phi}} f_2 + \frac{i}{r} \frac{\nu}{\sqrt{2}} (f_1 - f_3) = 0, \\
(4) \quad & \left(-\frac{\omega}{\sqrt{\Phi}} + i \sqrt{\Phi} \frac{d}{dr} + i \frac{\sqrt{\Phi}}{r} + i \frac{r}{\sqrt{\Phi}} \right) f_3 + \frac{i}{r} \frac{\nu}{\sqrt{2}} f_2 = 0.
\end{aligned} \tag{87}$$

Combining equations (2) and (4), instead of (87) we get

$$\begin{aligned}
(2) + (4), \quad & -\frac{\omega}{\sqrt{\Phi}}(f_1 + f_3) - i \left(\sqrt{\Phi} \frac{d}{dr} + \frac{\sqrt{\Phi}}{r} + \frac{r}{\sqrt{\Phi}} \right) (f_1 - f_3) = 0, \\
(2) - (4), \quad & -\frac{\omega}{\sqrt{\Phi}}(f_1 - f_3) - i \left(\sqrt{\Phi} \frac{d}{dr} + \frac{\sqrt{\Phi}}{r} + \frac{r}{\sqrt{\Phi}} \right) (f_1 + f_3) - \frac{2i}{r} \frac{\nu}{\sqrt{2}} f_2 = 0, \\
(3) \quad & -\frac{\omega}{\sqrt{\Phi}} f_2 + \frac{i}{r} \frac{\nu}{\sqrt{2}} (f_1 - f_3) = 0, \\
(1) \quad & \sqrt{\Phi} \left(\frac{d}{dr} + \frac{2}{r} \right) f_2 + \frac{1}{r} \frac{\nu}{\sqrt{2}} (f_1 + f_3) = 0.
\end{aligned}$$

It is easily verified that equation (1) is an identity when allowing for remaining ones. So independent equations are

$$\begin{aligned}
& -\frac{\omega}{\sqrt{\Phi}} f_2 + \frac{i}{r} \frac{\nu}{\sqrt{2}} (f_1 - f_3) = 0, \\
& -\frac{\omega}{\sqrt{\Phi}}(f_1 + f_3) - i \left(\sqrt{\Phi} \frac{d}{dr} + \frac{\sqrt{\Phi}}{r} + \frac{r}{\sqrt{\Phi}} \right) (f_1 - f_3) = 0, \\
& -\frac{\omega}{\sqrt{\Phi}}(f_1 - f_3) - i \left(\sqrt{\Phi} \frac{d}{dr} + \frac{\sqrt{\Phi}}{r} + \frac{r}{\sqrt{\Phi}} \right) (f_1 + f_3) - \frac{2i}{r} \frac{\nu}{\sqrt{2}} f_2 = 0.
\end{aligned} \tag{88}$$

Let us introduce new functions: $f = (f_1 + f_3)/\sqrt{2}$, $g = (f_1 - f_3)/\sqrt{2}$, then eqs. (88) look

$$\begin{aligned} f_2 = \frac{i\nu}{\omega} \frac{\sqrt{\Phi}}{r} g = 0, \quad -\frac{\omega}{\Phi} f - i\left(\frac{d}{dr} + \frac{1}{r} + \frac{r}{\Phi}\right) g = 0, \\ -\frac{\omega^2}{\Phi} g - i\omega\left(\frac{d}{dr} + \frac{1}{r} + \frac{r}{\Phi}\right) f + \frac{\nu^2}{r^2} g = 0, \end{aligned} \quad (89)$$

The system (89) is simplified by substitutions

$$\begin{aligned} g = \frac{1}{r\sqrt{1+r^2}} G(r), \quad f = \frac{1}{r\sqrt{1+r^2}} F(r), \\ f_2 = \frac{i\nu}{\sqrt{2}\omega} \frac{1}{r^2} G(r) = 0, \quad i\omega F = \Phi \frac{d}{dr} G, \\ i\omega \frac{d}{dr} F + \frac{\omega^2}{\Phi} G - \frac{\nu^2}{r^2} G = 0, \end{aligned} \quad (90)$$

So we have arrived at a single differential equation for $G(r)$:

$$(1+r^2) \frac{d^2 G}{dr^2} + 2r \frac{dG}{dr} + \left(\frac{\omega^2}{1+r^2} - \frac{\nu^2}{r^2} \right) G = 0. \quad (91)$$

In (91) let us introduce a new variable $z = -r^2$, which results in

$$4z(1-z) \frac{d^2 G}{dz^2} + 2(1-3z) \frac{dG}{dz} - \left(\frac{\omega^2}{1-z} + \frac{\nu^2}{z} \right) G = 0, \quad (92)$$

with the use of substitution $G = z^a(1-z)^b F(z)$. eq. (92) gives

$$\begin{aligned} 4z(1-z) \frac{d^2 F}{dz^2} + 4 \left[2a + \frac{1}{2} - (2a + 2b + \frac{3}{2})z \right] \frac{dF}{dz} + \\ + \left[\frac{4a^2 - 2a - \nu^2}{z} + \frac{4b^2 - \omega^2}{1-z} - 4(a+b)(a+b + \frac{1}{2}) \right] F = 0. \end{aligned} \quad (93)$$

With requirements

$$\begin{aligned} 4a^2 - 2a - \nu^2 = 0 \implies \\ a = \frac{1}{4} \pm \frac{1}{4} \sqrt{1 + 4\nu^2} = \frac{1}{4} \pm \frac{1}{2} (j + \frac{1}{2}) = -\frac{j}{2}, + \frac{j+1}{2}, \\ 4b^2 - \omega^2 = 0 \implies b = \pm \frac{\omega}{2}, \quad \omega > 0; \end{aligned} \quad (94)$$

to have solutions vanishing at $r = 0$ one must take positive values $a = (j+1)/2$, eq. (93) take the form

$$z(1-z) \frac{d^2 F}{dz^2} + \left[2a + \frac{1}{2} - (2a + 2b + \frac{3}{2})z \right] \frac{dF}{dz} - (a+b)(a+b + \frac{1}{2}) F = 0, \quad (95)$$

which is an equation of hypergeometric type

$$\gamma = 2a + \frac{1}{2}, \quad \alpha + \beta = 2a + 2b + \frac{1}{2}, \quad \alpha\beta = (a+b)(a+b + \frac{1}{2}),$$

that is

$$\alpha = a + b, \quad \beta = a + b + \frac{1}{2}, \quad \gamma = 2a + \frac{1}{2}. \quad (96)$$

To have polynomials one must take negative value for $b = -\omega/2$. So, parameters are

$$\alpha = \frac{j+1}{2} - \frac{\omega}{2}, \quad \beta = \frac{j+1}{2} - \frac{\omega}{2} + \frac{1}{2}, \quad (97)$$

and quantization is given by ¹:

$$\alpha = -n, \quad \omega_{n,j} = 2n + j + 1n, \quad (n = 0, 1, 2, \dots); \quad (98)$$

or in usual units $\omega = (c/\rho) (2n + j + 1)$; ρ is a curvature radius..

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¹ There exists symmetric variant $\beta = -n \implies \omega_{n,j+1} = 2n + (j+1) + 1$.

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